Robust FIR filter for Linear Discrete-time System

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Abstract: In this paper, a robust receding horizon finite impulse response (FIR) filter is proposed for a class of linear discrete time systems with uncertainty satisfying an integral quadratic constraint. The robust state estimation problem involves constructing the set of all possible states at the current time consistent with given system input, output measurements and the integral quadratic constraint.

Keywords: Robust, finite impulse response, receding horizon, integral quadratic constraint, set

1. Introduction

An estimation problem deals with recovering some unknown parameters or variables from measured information in physical or mathematical models. For the last 40 years, the estimation problem has been widely used in science and engineering areas.

Among estimation problems, a state estimator which is called the filter has been widely investigated and often combined with control design. In state estimation problems, the state variables are assumed to be unmeasurable and thus unknown.

In state space models with control inputs, filters for state estimation can have a finite impulse response (FIR) structure or an infinite impulse response (IIR). For the state \( x_k \) at time \( k \), the linear FIR filter without a priori information on the initial state can be represented in the following batch form:

\[
\hat{x}_{k|k} = \sum_{i=k-N}^{k-1} H_{k-i}y_i + \sum_{i=k-N}^{k-1} L_{k-i}u_i
\]

for some gains \( H_{k-i} \) and \( L_{k-i} \) [1], [2], [3]. The FIR filter has a similar form as (1) with \( k - N \) replaced by the initial time \( k_0 \). For IIR and FIR filters, initial states are denoted by \( x_{k_0} \) and \( x_{k-N} \), respectively. It is noted that the linear FIR filter (1) does not include an initial state term and its gains \( H_{k-i} \) and \( L_{k-i} \) are independent of initial state information, while the standard Kalman filter does include an initial state term and its gains are dependent of initial state information. As shown in (1), FIR filters utilize only finite measurements and inputs on the most recent time interval \([k - N, k]\) called the receding horizon or the horizon, which can avoid long processing time due to the large data sets in case of IIR filters when time increases. The FIR structure has inherent properties such as a bounded input-bounded output (BIBO) stability and robustness against temporary modeling uncertainties and round-off errors.

The above mentioned papers, however, did not consider the uncertain systems. But the filters require the robustness with respect to the model uncertainty. There are some attempts to investigate the robust Kalman filter [4], [5], [6], [7], [8], which needs the assumption of known initial state. In many cases, however, it may difficult to obtain the correct initial information because it may be costly and require many experience. Therefore, in the current paper, a new robust FIR filter will be investigated that is called the receding horizon robust FIR (RHRF) filter, which is independent on a priori information of the initial state. The RHRF filter represented in a batch form will be obtained by directly solving an optimization problem with the integral quadratic constraint.

The current paper is organized as follows. In Section 2, an RHRF filter with a batch form is proposed and finally the conclusions are stated in Section 3.

2. Receding Horizon Robust FIR Filter

Consider a linear discrete-time space state model:

\[
x_{k+1} = (A + G\Delta_{1,k}E_1)x_k + (B + G\Delta_{1,k}E_2)u_k,
\]

\[
y_k = (C + \Delta_{2,k}E_1)x_k + \Delta_{2,k}E_2u_k
\]

where \( x_k \in \mathbb{R}^n \) is the state, \( u_k \in \mathbb{R}^l \) and \( y_k \in \mathbb{R}^q \) are the input and measurement, respectively; \( A, B, C, G, E_1 \) and \( E_2 \) are given matrices with full row rank, \( \Delta_{1,k} \) and \( \Delta_{2,k} \) are uncertainty matrices satisfying

\[
\Delta_{1,k}^TQ\Delta_{1,k} + \Delta_{2,k}^TR\Delta_{2,k} \leq I,
\]

for all \( k \) and \( Q = Q' > 0, R = R' > 0 \).

Let

\[
w_k = \Delta_{1,k}(E_1x_k + E_2u_k),
\]

\[
v_k = \Delta_{2,k}(E_1x_k + E_2u_k).
\]

Then, the system represented by (2) and (3) is of the form:

\[
x_{k+1} = Ax_k + Bu_k + Gw_k,
\]

\[
y_k = Cx_k + v_k.
\]
The uncertainty \( w_k \) and \( v_k \) in (7) and (8) satisfies the following inequality called integral quadratic constraint:

\[
\sum_{i=k-N}^{k-1} \left( w_i^T Q w_i + v_i^T R v_i \right) = \sum_{i=k-N}^{k-1} \left( \Delta_{1,i}(E_1 x_i + E_2 u_i) \right)^T Q \left( \Delta_{1,i}(E_1 x_i + E_2 u_i) \right) + \left( \Delta_{2,i}(E_1 x_i + E_2 u_i) \right)^T R \left( \Delta_{2,i}(E_1 x_i + E_2 u_i) \right)
\]

\[
= \sum_{i=k-N}^{k-1} \left( E_1 x_i + E_2 u_i \right)^T \left[ \Delta_{1,i}^T Q \Delta_{1,i} + \Delta_{2,i}^T R \Delta_{2,i} \right] (E_1 x_i + E_2 u_i)
\]

\[
\leq \sum_{i=k-N}^{k-1} (E_1 x_i + E_2 u_i)^T (E_1 x_i + E_2 u_i)
\]

where the last inequality is obtained from (4).

The system (7) and (8) will be represented in a batch form on the most recent time interval \([k-N, k]\) called the horizon. The horizon initial time \( k-N \) will be denoted by \( k_N \) hereafter for simplicity. On the horizon \([k_N, k]\), the finite number of measurements is expressed in terms of the state \( x_k \) at the current time \( k \) as follows:

\[
Y_{k-1} = \hat{C}_N x_k + \hat{B}_N U_{k-1} + \hat{G}_N W_{k-1} + V_{k-1}
\]

where

\[
Y_{k-1} \triangleq [y_{k_N}^T \ y_{k+1_N}^T \ \cdots \ y_{k-1}^T]^T,
\]

\[
U_{k-1} \triangleq [u_{k_N}^T \ u_{k+1_N}^T \ \cdots \ u_{k-1}^T]^T,
\]

\[
W_{k-1} \triangleq [w_{k_N}^T \ w_{k+1_N}^T \ \cdots \ w_{k-1}^T]^T,
\]

\[
V_{k-1} \triangleq [v_{k_N}^T \ v_{k+1_N}^T \ \cdots \ v_{k-1}^T]^T,
\]

and \( \hat{C}_N, \hat{B}_N, \hat{G}_N \) are obtained from

\[
\hat{C}_1 \triangleq \begin{bmatrix} CA^{-1} \\ CA^{-1+i} \\ \vdots \\ CA^{-i+2} \end{bmatrix}, \quad \hat{B}_1 \triangleq \begin{bmatrix} CA^{-1}B \\ CA^{-2}B \\ \vdots \\ CA^{-i+1}B \end{bmatrix}
\]

\[
\hat{C}_i \triangleq \begin{bmatrix} \hat{C}_{i-1} \end{bmatrix} A^{-1}, \quad \hat{B}_i \triangleq \begin{bmatrix} \hat{B}_{i-1} \end{bmatrix} A^{-1} \quad \text{for } 2 \leq i \leq N.
\]

The noise term \( \hat{G}_N W_{k-1} + V_{k-1} \) in (10) can be shown to be zero-mean with covariance \( \Xi_{N} \) given by

\[
\Xi_{N} \triangleq \hat{G}_1 [\text{diag}(Q \ldots Q)] \hat{G}_1^T + [\text{diag}(R \ldots R)]
\]

With (13) and (14), the inequality (9) can be further represented by the following compact form:

\[
W_{k-1}^T Q_N W_{k-1} + V_{k-1}^T R_N V_{k-1} \leq X_{k-1}^T M_N X_{k-1} + X_{k-1}^T S_N U_{k-1} + U_{k-1}^T N_N U_{k-1}
\]

where

\[
X_{k-1} \triangleq [x_{k_N}^T \ x_{k+1_N}^T \ \cdots \ x_{k-1}^T]^T,
\]

\[
Q_i \triangleq [\text{diag}(Q \ldots Q)],
\]

\[
R_i \triangleq [\text{diag}(R \ldots R)],
\]

\[
M_i \triangleq [\text{diag}(E_1^T E_1 \ldots E_i^T E_1)],
\]

\[
S_i \triangleq [2 \cdot \text{diag}(E_2^T E_2 \ldots E_i^T E_2)],
\]

\[
N_i \triangleq [\text{diag}(E_2^T E_2 \ldots E_i^T E_2)].
\]

Let us define

\[
J[x_k, W_{k-1}] \triangleq \begin{bmatrix} W_{k-1}^T Q_N W_{k-1} + (K_{k-1} - \hat{C}_N x_k - \hat{G}_N W_{k-1})^T \\
\times R_N (K_{k-1} - \hat{C}_N x_k - \hat{G}_N W_{k-1}) - X_{k-1}^T M_N X_{k-1} \\
- X_{k-1}^T S_N U_{k-1} - U_{k-1}^T N_N U_{k-1} \end{bmatrix},
\]

where

\[
K_{k-1} = Y_{k-1} - \hat{B}_N U_{k-1},
\]

and

\[
X_{k-1} = \hat{A}_N x_k + \hat{B}_N U_{k-1} + \hat{G}_N W_{k-1},
\]

with

\[
\hat{A}_i \triangleq \begin{bmatrix} A^{-i} \\ A^{-i+1} \\ \vdots \\ A^{-1} \end{bmatrix}, \quad \hat{A}_i = \begin{bmatrix} A^{-i} \\ A^{-i+1} \\ \vdots \\ A^{-1} \end{bmatrix},
\]

\[
= \begin{bmatrix} \hat{A}_{i-1} \end{bmatrix} A^{-1}.
\]
\[
\begin{align*}
\hat{B}_i & = -A_i^{-1} B_i, \\
\hat{G}_i & = -A_i^{-1} G_i.
\end{align*}
\]

Arranging (26) in terms of \( W_{k-1} \) yields
\[
J[x_k, W_{k-1}] = W_{k-1}^{-1}(Q_N + \hat{G}_k^T R_N \hat{G}_N - \hat{G}_k^T M_N \hat{G}_N) W_{k-1} - 2W_{k-1}^T \times \left[ \hat{G}_k^T R_N (K_{k-1} - \bar{C}_N x_k) + \hat{G}_k^T M_N (\hat{A}_N x_k + \hat{B}_N U_{k-1}) + \hat{G}_k^T S_N U_{k-1} \right] + \mathbf{W}_{k-1},
\]
(32)
where \( \mathbf{W}_{k-1} \) is the terms without \( W_{k-1} \) and is as follows:
\[
\begin{align*}
\mathbf{W}_{k-1} &= (K_{k-1} - \bar{C}_N x_k) R_N (K_{k-1} - \bar{C}_N x_k) - (\hat{A}_N x_k + \hat{B}_N U_{k-1})^T M_N (\hat{A}_N x_k + \hat{B}_N U_{k-1}) - (\hat{A}_N x_k + \hat{B}_N U_{k-1})^T S_N U_{k-1} - U_{k-1}^T M_N x_k \\
&= x_k^T \left[ \hat{G}_k^T R_N \hat{C}_N - \hat{A}_N M_N \hat{A}_N \right] x_k - x_k^T 2\hat{G}_k^T R_N K_{k-1} \\
&+ 2\hat{A}_N^T M_N \hat{B}_N U_{k-1} + \hat{A}_N^T S_N U_{k-1} \bigg] + \left[ K_{k-1} R_N K_{k-1} - (\hat{B}_N U_{k-1})^T M_N (\hat{B}_N U_{k-1}) - (\hat{B}_N U_{k-1})^T S_N U_{k-1} - U_{k-1}^T N_N x_k \right] \\
&= x_k^T \Phi_1 x_k + 2x_k^T \Phi_2 + \Phi_3
\end{align*}
\]
with
\[
\begin{align*}
\Phi_1 & = \hat{C}_N R_N \hat{C}_N - \hat{A}_N^T M_N \hat{A}_N, \\
\Phi_2 & = (\hat{C}_N R_N \hat{B}_N - \hat{A}_N^T M_N \hat{B}_N - \frac{1}{2} \hat{A}_N^T S_N U_{k-1} \\
&- \hat{C}_N R_N \hat{B}_N - \hat{C}_N R_N \hat{B}_N - \frac{1}{2} \hat{A}_N^T S_N U_{k-1} \\
&- \hat{C}_N R_N \hat{B}_N - \hat{C}_N R_N \hat{B}_N - \frac{1}{2} \hat{A}_N^T S_N U_{k-1} \\
&- \hat{C}_N R_N \hat{B}_N - \hat{C}_N R_N \hat{B}_N - \frac{1}{2} \hat{A}_N^T S_N U_{k-1}
\end{align*}
\]
(33)

Let us define the following quantities for brevity:
\[
F = Q_N + \hat{G}_k^T R_N \hat{G}_N - \hat{G}_k^T M_N \hat{G}_N,
\]
(37)

Then, (32) can be rewritten as follows:
\[
J[x_k, W_{k-1}] = W_{k-1}^{-1} \mathbf{P} W_{k-1} - 2W_{k-1}^T (\hat{C}_N x_k) \Phi_1 x_k + 2x_k^T \Phi_2 + \Phi_3
\]
(39)

Now we are in a position to find out the corresponding set \( \chi_k[Y, U] \) of all possible states \( x_k \) at time \( k \) for the uncertain system represented by (7) and (8). If and only if there exists an uncertainty \( W_{k-1} \) such that
\[
J[x_k, W_{k-1}] \leq 0,
\]
(40)
then \( x_k \in \chi_k[Y, U] \).

Let consider the following minimization problem to obtain the set \( \chi_k[Y, U] \)
\[
\min_{W_{k-1}} J[x_k, W_{k-1}],
\]
(41)
As in [9], [10], (41) can be written as
\[
\min_{W_{k-1}} J[x_k, W_{k-1}] = (x_k - \tilde{x}_{k|k})^T \Pi^{-1} (x_k - \tilde{x}_{k|k}) - \rho(42)
\]
Recall that \( x^T P x + 2x^T q \) is minimized at \( x = -P^{-1} q \) and its optimal value is \( -q^T P^{-1} q \).

Therefore,
\[
J^* = \min_{W_{k-1}} J[x_k, W_{k-1}] = \min_{W_{k-1}} \left[ W_{k-1}^{-1} Q_N W_{k-1} + (K_{k-1} - \bar{C}_N x_k - \hat{G}_N W_{k-1})^T \times R_N (K_{k-1} - \bar{C}_N x_k - \hat{G}_N W_{k-1}) - X_{k-1}^T M_N X_{k-1} - X_{k-1}^T S_N U_{k-1} - U_{k-1}^T N_N x_k \right] \]
\[
= -P^{-1} F + \mathbf{W}_{k-1},
\]
(43)
when
\[
\begin{align*}
W_{k-1} &= \left[ Q_N + \hat{G}_k^T R_N \hat{G}_N - \hat{G}_k^T M_N \hat{G}_N \right]^{-1} \left[ \hat{G}_k^T R_N (K_{k-1} - \bar{C}_N x_k) + \hat{G}_k^T M_N (\hat{A}_N x_k + \hat{B}_N U_{k-1}) + \hat{G}_k^T S_N U_{k-1} \right] \\
&= \Psi_1 x_k - (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1})
\end{align*}
\]
(45)
the integral quadratic constraint.

Then the ellipsoid
\[
\hat{x}_{k|k} = \Phi_1 - \Psi_1^T P^{-1} \Psi_1
\]
and
\[
\hat{x}_{k|k} = \Phi_1 - \Psi_1^T P^{-1} \Psi_1
\]
are the solution to (47) and (48), respectively.

Therefore,
\[
J^* = \left[ x_k - (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1}) \right]^T P^{-1} \left[ x_k - (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1}) \right]
\]
\[
+ \Psi_3 U_{k-1} \right] + \hat{x}_k^T \Phi_1 x_k + 2 \hat{x}_k^T \Phi_2 + \Phi_3
\]
\[
= x_k^T \left[ \Phi_1 - \Psi_1^T P^{-1} \Psi_1 \right] x_k + 2 \hat{x}_k^T \left[ \Psi_1^T P^{-1} (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1}) \right]
\]
\[
+ \Phi_2 + \left[ (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1})^T P^{-1} (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1}) + \Phi_3 \right]
\]
\[
= (x_k - \hat{x}_{k|k})^T \left[ \Phi_1 - \Psi_1^T P^{-1} \Psi_1 \right] (x_k - \hat{x}_{k|k}) - \rho_k
\]
where
\[
\hat{x}_{k|k} = \left( \Phi_1 - \Psi_1^T P^{-1} \Psi_1 \right)^{-1} \left( \Psi_1^T P^{-1} (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1}) + \Phi_3 \right)
\]
and
\[
\hat{x}_{k|k} = \left( \Phi_1 - \Psi_1^T P^{-1} \Psi_1 \right)^{-1} \left( \Psi_1^T P^{-1} (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1}) + \Phi_3 \right)
\]
and
\[
\hat{x}_{k|k} = \left( \Phi_1 - \Psi_1^T P^{-1} \Psi_1 \right)^{-1} \left( \Psi_1^T P^{-1} (\Psi_2 Y_{k-1} + \Psi_3 U_{k-1}) + \Phi_3 \right)
\]
are the solution to (50) and (53), respectively.

Then the ellipsoid
\[
\chi_k = \left\{ x_k : (x_k - \hat{x}_{k|k})^T \Sigma^{-1} (x_k - \hat{x}_{k|k}) \leq \rho_k \right\}
\]
with
\[
\Sigma^{-1} = \Phi_1 - \Psi_1^T P^{-1} \Psi_1
\]
and
\[
\hat{x}_{k|k}, \rho_k
\]
in (50) and (53), respectively, is the set of all possible states at the current time consistent with given system input, output measurements and uncertainties satisfying the integral quadratic constraint.

3. Conclusion

In this paper, an integral quadratic constrained robust FIR filter is proposed for a class of linear discrete time uncertain systems. With the consideration of the system uncertainty, the proposed FIR filter has better performance than the existing FIR filter for the unstable systems.

References


